Review

In the first class we looked at the value today of future payments (introduction), how to value projects and investments. Present Value = Future Payment * Discount Factor. The discount factor is derived from the: discount rate, $\frac{1}{(1+r)}$.

Where r is the discount rate found from a comparative investment of similar period. Discount factor is the return we can get for a similar period. Government bond of similar time frame is a typical example (and also risk free).

More Risk = Lower r = Higher Discount Rate = less cost today for a risky future payment. The discount rate must compensate for the risk.

Valuing Long-Lived Assets

- If $100 today is worth more than $100 one year from today, then it stands to reason that $100 one year from today is worth more than $100 two years from today.

- The **Present Value of a Cash Flow** one year from today is:

- The present value of a cash flow two years from today is:

Discounting an investment which pays off many years from now. If the payout is in one year we know $PV(C_1) = C_1/(1+r)$. If two years $PV(C_2) = C_2/(1+r)^2$, we are discounting twice (squaring) because we are earning interest on interest, (1 + r)*(1 + r). This is the effect of compounding but in reverse, bringing the value back to the present.

Example

- After one year, $100 invested at 5% interest will be worth:

  FV = 100 * (1 + .05) = $105

- After two years, this investment will be worth:

$FV = 100 * (1 + .05)^2 = \$105$ and the present value would be

- Similarly, the present value of $110.25 received two years from today is:

$PV = \frac{110.25}{(1+.05)^2} = 100$ so we see we can go back to the present value as well.

Valuing Long - Lived Assets

- In general, the **Present Value** of a cash flow received **T years from today** is

$$PV(C_T) = \frac{C_T}{(1+r)^T}$$

(Present value of C which is a payment at time T)

This equation gives us the present value of any payment C at time T.

- And the **Future Value at time T** of a present amount of money is:

$$FV(C_0) = C_0 \times (1+r)^T$$

(A present amount C0 will earn interest at rate r over T periods.)

- **Compounding** an investment **m times a year for T years** provides for **Future Value** of wealth:

$$FV = C_0 \times \left(1 + \frac{r}{m}\right)^{m \times T}$$

(Future value of the C we have at time 0)

MOST IMPORTANT TO CONSIDER WHERE IN TIME YOUR PAYMENT LIES.
LOOK AT THE C SUBSCRIPT, IS IT C_0 OR C_T ?

Compounding m times per year: $C_0(1+\frac{r}{m})^m$

Continuous Compounding: $C_0 e^{rT}$

Where r is the annual interest rate. (could probably be rate per period)

Example

- Suppose you can choose one of the following prizes:

  - $1,000 now
  - $1,500 four years from now

- Assume that the discount rate is 10%

- Which prize is more valuable?

$$PV_{10} = \$1{,}000$$

$$PV_{15} = \frac{1500}{(1+.1)^4} = \$1{,}024.52 \quad \Leftarrow \quad \text{More Valuable}$$

So getting $1,500 in 4 years is the same as getting $1,024.52 today.
SAME THING !

We can also consider the Future Value of $1,000 …

$$FV_{10} = 1000(1+.1)^4 = \$1{,}464.10$$

Still we see that the $1,500 option is better.

Valuing cash flows over many periods

- A **stream of cash flows** over many periods can be valued as:

$$PV = \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \frac{C_3}{(1+r)^3} + \cdots \quad \frac{C_T}{(1+r)^T}$$

(The sum of each individual cash flow.)

- This formula is called the **Discounted Cash Flow (DCF)** Formula
- A shorthand way to write this formula is

$$PV = \sum_{t=1}^{T} \frac{C_t}{(1+r)^t}$$

Net Present Value

- The **Net Present Value (NPV)** of an investment project is calculated by subtracting the required investment from the present value of the future cash flows generated by the investment

$$NPV = -Cost + PV$$

- For **one period** case

$$NPV = C_0 + \frac{C_1}{(1+r)}$$

- C_0 will usually be negative since it reflects an investment, which is a cash outflow (C_0 is the initial investment).

NPV Example

- Suppose you can invest $950 today and earn a certain cash flow of $1,000 one year from today. The risk-free rate is 5%. What is the net present value of this investment?

NPV RULE SAYS TAKE ANY PROJECT WITH POSITIVE NPV.
NPV RULE IS THE BEST GUIDE AVAILABLE.

$$NPV = -950 + \frac{1000}{(1+.05)^1} = -950 + 952.38 = \$2.38$$

Should we take this project?
Result is positive so project is good!

Present Value is the value in current terms of a future payment. Not taking this project is the same as losing $2.38. To say that you are not satisfied with the $2.38 and that you think you could make a better investment means that you are not using the correct discount rate. All risk, effort, anything else effecting this project is built into the discount rate!

Calculating NPV for Multi-Period Investment Projects

- Just as in the one period case, we calculate the net present value of a multi-period investment project by adding the (usually negative) first period cash flows

$$NPV = C_0 + \sum_{t=1}^{T} \frac{C_t}{(1+r)^t} = \sum_{t=0}^{T} \frac{C_t}{(1+r)^t}$$

(We rewrite the summation equation to include the payment at time 0)

First term will be $\frac{C_0}{(1+r)^0} = C_0$

Example

- An investor has the following stream of cash flows

Period	0	1	2	3	4
Cash Flow	(1,000)	400	500	450	200

- What is the NPV of this investment if the discount rate is 10%? 25%?

$$NPV = \frac{-1000}{1.1^0} + \frac{400}{1.1} + \frac{500}{1.1^2} + \frac{450}{1.1^4} + \frac{200}{1.1^4} = \$251.55$$

At 10% the NPV is positive so take the project.

$NPV_{25\%}$ = -47.68 BAD PROJECT!

Higher discount rate decreases a future payment. A current value is not effected by discount rate. Thus, the negative initial payment can play a big role when evaluating different discount rates.

Looking for Shortcuts

- A **Perpetuity** is a fixed cash flow, C, each year which last forever. A constant stream of cash flows without end. The value of C is the same for each payment over the entire life.

$$PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \ldots$$

The series can be shown to converge (see text).

- The formula for the **Present Value of a Perpetuity** is:

$$PV = \frac{C}{r}$$

r = interest rate in percent
C = payment to occur each year forever

Example

- You just won a lottery that will pay you $100,000 per year forever. The lottery organizers offered to pay you $1 million in cash today instead of the perpetuity. The interest rate is 8%. Should you accept?

This is a perpetuity so we use:

$$PV = \frac{C}{r} = \frac{100000}{.08} = \$1.25 \text{ Million} > \$1 \text{ Million}$$

We find that we can sell the perpetuity for it's present value of $1.25 Million which is greater than the $1 million buyout offer. We want to chose the option with the highest present value.

IN THIS CLASS EVERYTHING IS SELLABLE AND EVERYTHING IS BUYABLE!

Looking for Shortcuts

- A **Growing Perpetuity** is a cash flow that grows with a constant rate, g, each year forever. The power of the g term is one less than the power of the r term. Growth must begin in the second term and first payment occurs one period in the future.

$$PV = \frac{C}{1+r} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \ldots$$

- The formula for the **Present Value of a Growing Perpetuity*** is:

$$PV = \frac{C}{r-g}$$

where g < r

* **3 Important points concerning the above formula** (text pg. 78):

1. The Numerator is the cash flow ONE PERIOD IN THE FUTURE, not the cash flow at time 0.
2. The Interest Rate and the Growth Rate: the interest rate r must be greater than the growth rate g. As the magnitude of g approaches r the present value will become infinite. The present value is undefined for r = g.
3. Timing Assumption: the formula assumes that cash flows are received and distributed at regular intervals. In practice, cash flows both into and out of real-world firms both randomly and nearly continuously.

g = growth rate per period in percent
r = interest rate in percent
C = payment to occur at the *end* of the *first* period

If we have **Growth** in the **First Period** the equation becomes:

$$PV = \frac{C(1+g)}{(1+r)}$$

Example

- You own a patent on the technology used to make ink for ballpoint pens. The royalties on this patent will be $750,000 this year, and will grow at 4% per year forever. The interest rate is 10%. What is the value of your patent?

$$PV = \frac{C}{r - g} = \frac{750{,}000}{.1 - .04} = \$12.5 \text{ Million}$$

Were we to sell the patent this would be the (minimum) price.

Looking for Shortcuts

- An **Annuity** is an asset that pays a fixed amount each period for a specified number of periods

$$PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \ldots + \frac{C}{(1+r)^T}$$

- The formula for the **Present Value of an Annuity*** is (4.13 on pg 80):

$$PV = \frac{C}{r}\left(1 - \frac{1}{(1+r)^T}\right)$$

Annuity lasting T periods, C is the same value for each payment.

* This formula is only valid for **Annuity in ARREARS** meaning the first annuity payment occurs one full period (typically year) in the future.

r = interest rate in percent
C = payment to occur at the end of each period
T = number of periods

Example

- You want to buy a house. You have $50,000 for the down payment and can afford to pay $2,000 per month on the mortgage. The 30-year mortgage interest rate is 9% per year, or 0.75% per month. What is the most expensive house that you can buy?

T = 30 C = $2,000 r = .09/12 = .75%

$$PV = \text{Downpayment} + \frac{C}{r}(1 - \frac{1}{(1+r)^T}) = 50{,}000 + \frac{2000}{.0075}(1 - \frac{1}{(1+.0075)^{30}})$$

Looking for shortcuts

- A **Growing Annuity** is an asset that pays an amount that is growing with a constant rate, g, each period for a specified ***finite*** number of periods. The power of the g term is one less than the power of the r term. Growth must begin in the second term and first payment occurs one period in the future. (growth begins in second year)

$$PV = \frac{C}{1+r} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \ldots + \frac{C(1+g)^{T-1}}{(1+r)^T}$$

- The formula for the **Present Value of a Growing Annuity** is:

$$PV = \frac{C}{r-g}\left(1-\left(\frac{1+g}{1+r}\right)^T\right)$$

g < r, must be getting amount C at end of first period, growth begins in second year.

g = growth rate per period in percent
r = interest rate in percent
C = payment to occur at the end of the first period
T = number of periods

Note that C, the value of the payment, can be an expression such as C(1+z). Just replace all C's with the expression (such as C(1+z)).

Example

- You are evaluating an income property that is providing increasing rents. Net rent is received at the end of each year. The first year's rent is expected to be $8,500 and rent is expected to increase 7% each year. What is the present value of the estimated income stream over the first 5 years if the discount rate is 12%?

T = 5 C = $8,500 r = 12 g = 7%

$$PV = \frac{8500}{.12 - .07}(1 - (\frac{1+.07}{1+.12})^5) = \$34,706$$

Nominal and Effective Interest Rates (EAR)

- **Effective Annual Interest Rate (EAR)** is the actual rate generated by the investment. It results from adjusting the nominal annual interest rate to compounding during the year

$$r_{effective} = \left(1 + \frac{r_{nominal}}{m}\right)^m - 1$$

where **m is the number of compound intervals per year**

- Given a constant nominal rate, the effective rate increases with **m** (number of compounding periods per year)
- If **m** becomes very large (infinite), then

$$r_{effective} = e^{r_{nominal}} - 1 \qquad \text{where} \quad e \approx 2.71$$

MUST KNOW e VALUE FOR EXAM

Continuous compounding

Difference between nominal and effective interest rates: want to compare investments but sometimes return is more often than once a year. Say we have a return of 10% per year paid twice a year. We will earn more than the 10% but how much more? Semi-Annual rate would be $1 + \frac{0.1}{2} = 1.05\%$. Now we have 100*1.05=105 after 6 months and then 105*1.05=110.25 total return after 1 year. So the effective interest rate over a year is $r_{eff} = (1 + \frac{0.1}{2})^2 - 1 = 10.25\%$.

Moving our payment closer to the present means greater value, we can use the time value of money to make more money!

Texas Instruments BAII	
[2nd] [ICONV]	Opens interest rate conversion menu
[↑] [C/Y=] 12	Sets 12 payments per year
[↓][NOM=] 18 [ENTER]	Sets 18 APR.
[↓] [EFF=] [CPT] 19.56	

Example

- Which investment would you prefer?

- An investment paying interest of 12% compounded annually?
- An investment paying interest of 11.75% compounded semiannually?
- An investment paying interest of 11.5% compounded continuously?

Annual

R = 12%

Semiannual

$$r_{eff} = (1 + \frac{.1175}{2})^2 - 1 = 12.09\%$$

11.75% compounded semiannually slightly better than 12% annually.

Continuous

$$r_{eff} = e^{.115} - 1 = 12.187\%$$

11.5% compounded continuously gives the greatest return even though it has the lowest nominal rate. The reason is the continuous compounding rate.

Chapter Summary

1. Two basic concepts, *future value* and *present value*, were introduced in the beginning of this chapter. With a 10-percent interest rate, an investor with $1 today can generate a future value of $1.10 in a year, $1.21 [$1 × $(1.10)^2$] in two years, and so on. Conversely, present-value analysis places a current value on a later cash flow. With the same 10-percent interest rate, a dollar to be received in one year has a present value of $0.909 ($1/1.10) in year 0. A dollar to be received in two years has a present value of $0.826 [$1/$(1.10)^2$].
2. One commonly expresses the interest rate as, say, 12 percent per year. However, one can speak of the interest rate as 3 percent per quarter. Although the stated annual interest rate remains 12 percent (3 percent × 4), the effective annual interest rate is 12.55 percent [$(1.03)^4 - 1$]. In other words, the compounding process increases the future value of an investment. The limiting case is continuous compounding, where funds are assumed to be reinvested every infinitesimal instant.
3. A basic quantitative technique for financial decision making is net present value analysis. The net present value formula for an investment that generates cash flows (C_i) in future periods is

$$\text{NPV} = -C_0 + \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_N}{(1+r)^N} = -C_0 + \sum_{i=1}^{N} \frac{C_i}{(1+r)^i}$$

   The formula assumes that the cash flow at date 0 is the initial investment (a cash outflow).

4. Frequently, the actual calculation of present value is long and tedious. The computation of the present value of a long-term mortgage with monthly payments is a good example of this. We presented four simplifying formulas:

Perpetuity: $\text{PV} = \frac{C}{r}$

Growing perpetuity: $\text{PV} = \frac{C}{r-g}$

Annuity: $\text{PV} = C\left[\frac{1}{r} - \frac{1}{r \times (1+r)^T}\right]$

Growing annuity: $\text{PV} = C\left[\frac{1}{r-g} - \frac{1}{r-g} \times \left(\frac{1+g}{1+r}\right)^T\right]$

5. We stressed a few practical considerations in the application of these formulas:
   a. The numerator in each of the formulas, C, is the cash flow to be received *one full period hence*.
   b. Cash flows are generally irregular in practice. To avoid unwieldy problems, assumptions to create more regular cash flows are made both in this textbook and in the real world.
   c. A number of present value problems involve annuities (or perpetuities) beginning a few periods hence. Students should practice combining the annuity (or perpetuity) formula with the discounting formula to solve these problems.
   d. Annuities and perpetuities may have periods of every two or every *n* years, rather than once a year. The annuity and perpetuity formulas can easily handle such circumstances.
   e. One frequently encounters problems where the present value of one annuity must be equated with the present value of another annuity.

Practice questions

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4	4.17	
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8	4.34	
9	4.36	
10	4.41	
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