

The Theory of the Firm

Economic Markets

We've discussed demand, from the theory of a consumer. For supply we will examine the firms perspective, what inputs should they use, what are their long run cost functions, what form do they take? This all leads to aggregate supply, how firms react to different prices in the market.

We will think of the firm in terms of a production function. If a firm is producing widgets, to product q widgets it's going to combine various levels of inputs. We will keep our analysis simple (as we did in demand), were going to think of production as coming from two inputs. These inputs can be anything, raw material and labor for example. But traditionally the way we look at these things is to think of the two inputs as being **labor** and **capital**. But they could be anything and we can generalize this analysis to many inputs.

Basic Concepts

- The Production Function
  - The firm's **production function** for a particular **good (q)** shows the maximum amount of the good that can be produced using alternative combinations of **capital (k)** and **labor (l)**

$$q = f(k, l)$$

Here we are describing the firm as a production function with variables capital, k, and labor, l allowing us to produce q goods. We will be interested in understanding how firms can change the different levels of inputs and what impact that has on the units produced. We will then think about what the firm is going to maximize. If it wants to produce so many widgets, what is the optimal level of the inputs, how much capital does it need and how much labor. Will also ask, if a firm wishes to maximize it's profits, how many units should it produce in order to do so. How much does it cost to create these widgets? Similar to the analysis we did last week but will be some differences.

Product Curves, cont'd

❑ Marginal Product

- To study variation in a single input, we define **marginal product** as the additional output that can be produced by employing one more unit of that input while holding other inputs constant
- For example, the marginal product of capital is given by:

$$\text{CAPITAL:} \quad MP_k = \frac{\partial q}{\partial k} = f_k$$

Marginal Product, if we vary one of the inputs, if we increase the amount of labor that we use, how does it change the number of units that we create? So **holding one of the inputs constant**, if we change one of the other inputs how much does it change the output?

We have a certain fixed level of capital, if I increase my labor hours from 1000 to 1001 how many more units can I product. This is what we are interested in when we are talking about marginal product. How much more can I produce given one input increased and the other held constant?

Example, if we focus on capital, the Marginal Product of capital if simply the partial derivative of the number of units produced, q, wrt the level of capital **(we denoting this as f_k)**. We are looking at the rate of increase of the number of units given an increase in the level of capital holding labor constant. We derive the labor version of marginal product by taking the partial derivative wrt labor holding capital constant:

$$\text{LABOR:} \quad MP_l = \frac{\partial q}{\partial l} = f_l$$

Marginal Product, cont'd

- **Diminishing Marginal Productivity**
  - The marginal product of an input depends on how much of that input is used
  - In general, we assume diminishing marginal productivity
  - Using labor as an example, this would mean:

$$\frac{\partial MP_l}{\partial l} = \frac{\partial^2 f}{\partial l^2} = f_{ll} < 0$$

We usually assume there is diminishing marginal productivity meaning as we add more and more of one input while holding the other input fixed we get less bang for the buck. If I have 50 people working in the factory and only so much capital and I add one more person I will increase my outputs. As I add the 52 person I'm going to increase the output but it will do so at a diminishing rate (will not see as must as an increase as we did when we added number 51). So adding additional inputs holding the other input fixed is going to increase the output but the rate of change is diminishing, less and less benefit by adding those factors. This is what is meant by **Diminishing Marginal Productivity**.

What does this physically mean? Its means that the second partial derivative wrt that input is negative.

Diminishing Marginal Productivity, cont'd

- Because of diminishing marginal productivity, 19th century economist Thomas Malthus worried about the effect of population growth on labor productivity
- But changes in the marginal productivity of labor over time also depend on changes in other inputs such as capital
  - We need to consider f_{lk} which is often > 0

Our model holds the other factors constant, if the other factors can change then it may be possible to get more productivity out of labor because the derivative will be wrt more than one term, for instance the partial derivative wrt capital *and* the partial derivative wrt labor.

Getting more from the labor we are employing as we add other things including computer systems and management.

Product Curves, cont'd

❑ Average Product

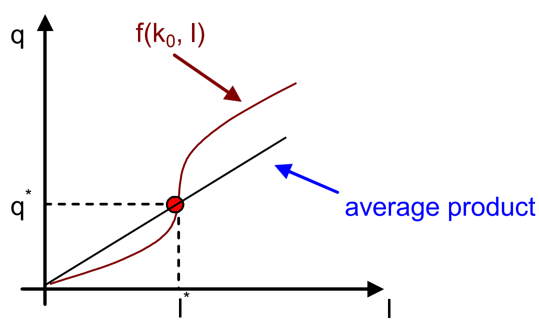
- The average product of an input, say labor, is the total product produced divided by the quantity of labor used in its production

$$AP_l = \frac{q}{l} = \frac{f(k,l)}{l}$$

- Note that AP_l also depends on the amount of capital employed

Take a particular number of outputs, say q = 1000 widgets. And I look at the amount of labor or capital that was necessary to produce that output, the average output is simply the number of units divided by the inputs employed. We're looking at the labor example, could just as easily be capital.

We see that AP is a function of the other input, in this case capital. So dependent on the amount of capital we have employed, the average product can vary even with constant levels of labor. If I change the amount of capital, even with the same amount of labor I can have varying amounts of average product.



Suppose we fix capital at k_0. Want to start with no labor (origin) the increase the amount of labor (combined with this fixed level of capital) and we will see an increase in the level of output. Trying to map the amount of units I'm producing if I increase labor holding capital fixed, $f(k_0, l)$. Now we will calculate AP at a particular point on the curve by drawing a line through that point and calculating it's slope. At this point we are producing q units with l hours of labor, so the average is just q/l (since the line is through the origin, no offset).

Output Elasticity

Output Elasticity: if I'm interested in knowing if I can generate a small change in a quantity that I produce divided by the starting level of production, say I'm at 1000 units and I make a change, this is going to give a percentage change in the number of units that I produce, and I want to compare that to a change in one of the input factors that generates this change in units. Late the second change be labor, if I change the number of labor hours by a small number divided by the starting level of labor I will get the percentage change in units produced divided by the percentage change of labor employed HOLDING CAPITAL K FIXED.

$$\left.\frac{\frac{\Delta q}{q}}{\frac{\Delta l}{l}}\right|_{\text{holding k fixed}} = \frac{1}{\mathrm{q}}\frac{\delta q}{\delta l} = \frac{1}{\mathrm{q}}\mathrm{F_l} \qquad \text{where} \qquad \frac{\delta q}{\delta l} = \mathrm{F_l}$$

For example, if I increase my labor hours by 1%, how many percentage point increase do I get in the number of units that I produce? We could also derive this calculation for capital holding labor constant.

We take the delta q and l ratios to their limit which gives us the starting labor over the starting quantity, l/q, multiplied by the partial derivative of the number of units produced wrt labor. This is the output elasticity of labor. (we can do the same for capital). We know from the previous slides that

$$\frac{1}{\mathrm{q}}\mathrm{F_l} = \frac{\mathrm{MP_l}}{\mathrm{AP_l}}$$

This is the ratio of Marginal Product Labor to Average Product Labor. Output Elasticity is the Marginal Product divided by the Average Product.

When the two are equal a 1% increase in labor will lead approximately to a 1% increase in output.

$$\mathrm{MP_l} = \mathrm{Fl} \qquad \mathrm{AP_l} = \frac{\mathrm{q}}{\mathrm{l}}$$

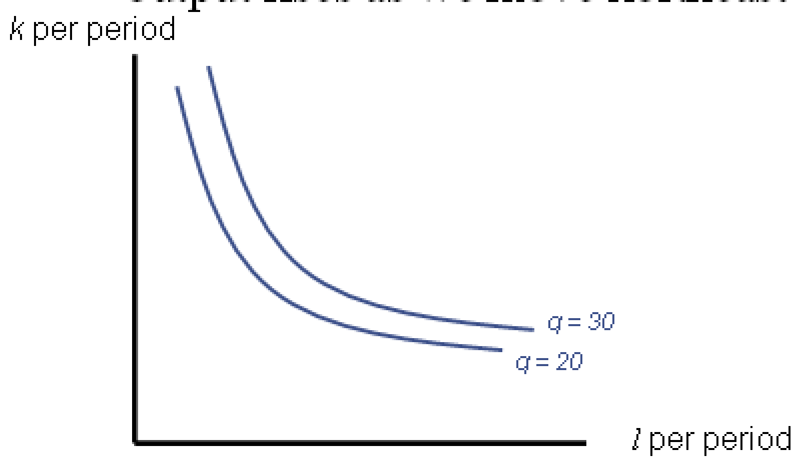
Notice it is possible for $\mathrm{MP_l} < 0$ for some levels of l. This would mean we are generating less output for the one unit of labor added. But this is possible, for instance a factory running at 100% production fully staffed with workers. If I add more workers to that factory but don't actually change the amount of capital assets they operate they actually end up just getting in each others way. The firm becomes less efficient. In this way we can get diminishing returns by adding more labor.

Basic Concepts, cont'd

- Isoquants, this is a form of indifference curve. (a curve that represents all the possible efficient combinations of inputs that are needed to produce a quantity of a particular output or combination of outputs)
  - To illustrate the possible substitution of one input for another, we use an isoquant map
  - An isoquant shows those combinations of k and l that can produce a given level of output (q_0)

We are looking at different levels of the two inputs and seeing when they give us a fixed level of outputs.

- Each isoquant represents a different level of output
  - output rises as we move northeast

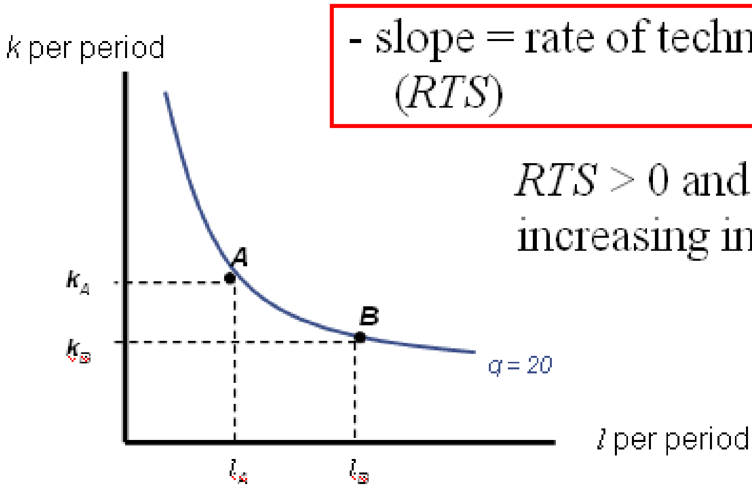


They are usually assumed to look something like this. Usually assumed to be concave to the origin (not necessary, can think of cases where adding more gives you less utility). In the same way as utility was increasing for the consumer, the same is usually true for the isoquants. The more labor or capital we have (the more we move in a northeast direction away from the origin), the higher our level of production. Anywhere on a particular line we have a fixed level of production, q, and the axis are the different combinations of labor and capital that can give us these production outputs. The more labor and capital we have the higher the overall output.

Where we had RCS in the supply case we have **Rate of Technical Substitution (RTS)** in this case. The idea here is in order to remain at the same level of output, as we move down the curve how much labor has to be replaced by capital to keep the output constant. If I reduce my labor how much more capital do I need in order to produce the same amount? This is called the RTS. It is derived as the negative slope of the Isoquant line at any particular point. So this means it will varies depending on the amounts of capital and labor and their slope on the isoquant at any particular point.

❑ Rate of Technical Substitution (RTS)

* The slope of an isoquant shows the rate at which l can be substituted for k



- slope = rate of technical substitution (*RTS*)

RTS > 0 and is diminishing for increasing inputs of labor

RTS = - Slope

It is usually assumed that RTS > 0. Assume that the tangent is downward sloping, and because we assume that the isoquants are concave to the origin, there is usually a diminishing effect, the slope flattens out the further you go down the curve.

How do we measure RTS? Same idea as before, the rate of change of k wrt l. That will give us the slope of that line.

❑ The **rate of technical substitution (*RTS*)** shows the rate at which labor can be substituted for capital while holding output constant along an isoquant

$$RTS\left(l \text{ for } k\right) = -\left.\frac{dk}{dl}\right|_{q=q_0}$$

Where k is a function of l, the slope of that line is given by the derivative of k wrt l **fixing at a certain level of output**.

On the next page we take this a step further (similar to the consumer analysis we saw last class)…

Rate of Technical Substitution, cont'd

- RTS and Marginal Productivities
  - Take the total differential of the production function:

$$dq = \frac{\partial f}{\partial l} \cdot dl + \frac{\partial f}{\partial k} \cdot dk = MP_l \cdot dl + MP_k \cdot dk$$

Here we have the rate of change of output wrt l (considering that q=f(k, l)). We also have the rate of change of output wrt k. And each is multiplied by a small change in their respective value (dl and dk).

We assume that we are keeping q constant, holding on the same isoquant. So we will set dq equal to zero.

$$MP_l \cdot dl + MP_k \cdot dk = 0$$

Now we can rearrange the terms…

$$MP_l \cdot dl = -MP_k \cdot dk$$

And we arrive at …

  - Along an isoquant dq = 0, so

$$RTS\left(l \text{ for } k\right) = -\left.\frac{dk}{dl}\right|_{q=q_0} = \frac{MP_l}{MP_k}$$

Basic Concepts,
Different Forms of the Production Function

- Shape of the Production Function
  - The Linear Production Function
  - Fixed Proportions
  - Returns to Scale, if I increase both of my inputs by proportion does my output change bu the same proportion?

The **Linear Production Function**

- Suppose that the production function is

  $$q = f(k,l) = ak + bl$$

- This production function exhibits constant returns to scale

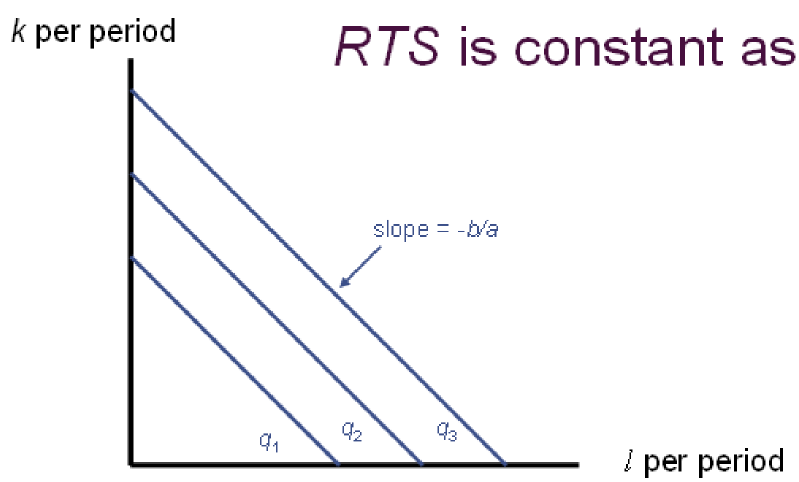
  $$f(tk,tl) = atk + btl = t(ak + bl) = tf(k,l)$$

- All isoquants are straight lines
  - RTS is constant

The Linear Production Function, cont'd

Capital and labor are perfect substitutes

RTS is constant as *k*/*l* changes



Notice that this production function exhibits constant returns to scale. If I increase the amount of capital I have, say I double it, if I increase the amount of labor I have, say I double that as well, [increasing by a constant amount] if the Production Function is LINEAR then it will increase the production by the same amount.

For example, if I increase labor and capital each by 10% then overall production will increase by 10%. Constant Returns to Scale. Only for this type of function and only because it is linear. In general this will not be true. Most industries will see increased output but as the business becomes larger the increases will fall off.

With these types of functions the Isoquants are also straight lines.

Capital and Labor are perfect substitutes. I can get the same level of production if I reduce my labor by adding capital in a fixed proportion. What is that fixed proportion? **It is given by the ratio of the constants b/a (the slope of the line).** I can produce a certain level of output by using all labor or all capital or combinations of the two.

Rate Technical Substitution is constant. It is the same where ever we are on the particular isoquant.

Fixed Proportions, the other extreme.

- Suppose that the production function is

  $q = \min(ak, bl) \quad a,b > 0$

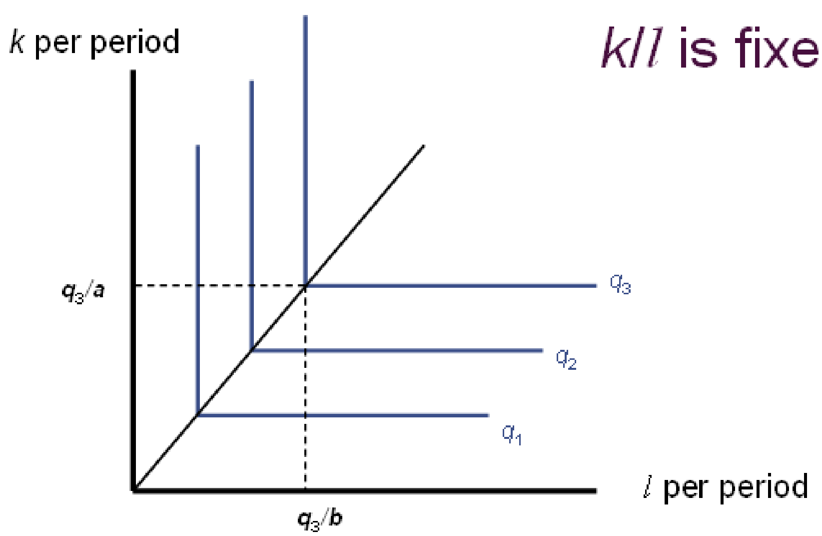
- Capital and labor must always be used in a fixed ratio
  - the firm will always operate along a ray where k/l is constant

I can use capital or labor or ant mixture of the two. But the amount of output I get Is limited to the minimum of these two values, q=min(ak, bl). So it's kind of an idea that in order to produce something I need a certain amount of capital to go with my labor. If I add capital over and above that I get no benefit.

Fixed Proportions

No substitution between labor and capital is possible

k/l is fixed at *b/a*



If I have a fixed amount of labor then I need a certain amount of capital to produce a certain number of units. If I increase my capital above that level I get no further benefit, it's just money down the drain. If I have a certain level of capital I need a certain level of labor to produce a given number of units, increasing labor gives no additional benefit.

The last page is telling us that we need the mix of capital and labor right at the sweet spot, no benefit from going over.

Fixed Proportions and Linear Production are two extremes, most firms are somewhere in-between.

The ratio of capital to labor is fixed by the ration b/a.

Returns to Scale

- How does output respond to increases in all inputs together?
  - suppose that all inputs are doubled, would output double?
- Returns to scale have been of interest to economists since the days of Adam Smith

We've already seen this from the linear production function, where we had constant returns to scale. In that case if we double the inputs we would double the outputs. In general this will not be the case. If I add 10% to my inputs I'm not necessarily going to see a 10% increase in my outputs. I may see more or less return.

What will lead to increased or decreased returns to scale? Adam Smith focused on two things? Adam Smith focused on two things:

- Division of Labor
- Specialization

Adam Smith studies people who make pin. He noticed that when they manufactured a small order each laborer had to manufacture each part of the pin (the entire pin). As production got higher labor could specialize, divide the job amongst themselves. As the size of the production increased it allowed specialization and division of the labor. They became more efficient as a result. Therefore as the size of the firm increased the firm became more efficient. Adding additional labor actually gave more output than would be expected by returns of scale. On the flip side however the larger you become the more difficult it becomes to manage the operation. Therefore adding additional labor after a certain point the firm became less efficient. It is possible to see lower returns than returns to scale.

So it is possible for a single business to initially see greater output and greater returns to scale and then later on as it becomes too large see lower than returns to scale. This was Adam Smith's discovery. This theory holds up under examination but it depends on the type of industry you are in. If your in an industry that is very specialized then the economies of scale last a lot longer. A good example is beer manufacturers and wine manufacturers. It is much easier for beer to be produced in bulk then it is for wine to be produced in bulk. It has to do with economies of scale and diseconomies of scale.

Suppose I have the production function, it being a function of labor and capital. Then if I increase my inputs by the same amount, say 10%, if I get the same proportional output in my output, then I describe that as **CONSTANT RETURNS TO SCALE**. The linear production function exhibits constant returns to scale. If I see less output for a given percentage increase in the input, I increase all my inputs by 10% and I get less than 10% increase in my outputs, we woul call this **DECREASING RETURNS TO SCALE**. The opposite is the case where I double my inputs and I get MORE than double my output, this is called **INCREASING RETURNS TO SCLAE**.

EXAMPLES:

Knob-Dopler

$q = A(k^{\alpha} l^{1-\alpha})$ where $0 < \alpha < 1$

increase k to tk and l to tl where t>0

$$q(tk, tl)=A(t^{\alpha} k^{\alpha})(t^{1-\alpha} l^{1-\alpha}) = A(t^{\alpha} t^{1-\alpha})(k^{\alpha})(l^{1-\alpha})$$

$$=tAk^{\alpha} l^{1-\alpha} = tq(k,l).$$

Constant return to scale. This is an example of a function which is not linear but also gives constant return to scale.

CES Production Function

$$q(k,l) = A(k^r + l^r)^{1/r}$$

$$q(tk, tl)=A(t^r k^r + t^r l^r)^{1/r} = tq(k,l)$$

This one is also constant return although it in no way looks liner.

Production Function

when $q(k, l)=k + l + kl$

$q(tk, tl)=tk + tl + t^2 kl$

if $t>1$, $tq(k,l) \neq q(tk,tl)$

This last example is a classic example of a non-constant return to scale. Is this one diminishing or increasing?

This would be an increasing return to scale if t>1.

If t<1 this would be diminishing.

To consider a real world example, I open up my own store, initially I make a profit from the one store. Say I work in Hartford, I may be able to open 4 stores and my profits grow very quickly because I can manage those 4 stores very easily so I get increased return. Now say I open another group of stores in Ohio, I loss the ability to manage these things hands on, I have to hire others, another layer of management. May see the returns diminish, I don't get so much additional profit by adding additional stores.

In manufacturing when I am small I can specialize, I can pick the jobs I want to run. As I become larger I have to have set mechanisms, focus on selling a certain type of output. I can't be as flexible as I was. Therefore, I may initially see my output increase rapidly but later see my output decline.

When we speak of diminishing we are not saying the output declines, it just stops increasing at the same rate.

Returns to Scale

- Smith identified two forces that come into operation as inputs are doubled

1. greater division of labor and specialization of function
2. loss in efficiency because management may become more difficult given the larger scale of the firm

- If the production function is given by q = f(k,l) and all inputs are multiplied by the same positive constant (t >1), then

Effect on Output	Returns to Scale
$f(tk,tl) = tf(k,l)$	Constant
$f(tk,tl) < tf(k,l)$	Decreasing
$f(tk,tl) > tf(k,l)$	Increasing

←same proportional output

←less output for given increase

← double inputs give more than double outputs

Returns to Scale

- It is possible for a **production function** to exhibit constant returns to scale for some levels of input usage and increasing or decreasing returns for other levels

Economists refer to the degree of returns to scale with the implicit notion that only a fairly narrow range of variation in input usage and the related level of output is being considered

Basic Concepts, cont'd

- The **elasticity of substitution (σ)** measures the proportionate change in k/l relative to the proportionate change in the RTS along an isoquant

$$\sigma = \frac{\%\Delta(k/\mathbf{l})}{\%\Delta RTS} = \frac{d(k/\mathbf{l})}{dRTS} \cdot \frac{RTS}{k/\mathbf{l}} = \frac{\partial \ln(k/\mathbf{l})}{\partial \ln RTS}$$

- The value of σ will always be positive because k/l and RTS move in the same direction

What we are looking at here is if we change the ratio of two inputs then how does our ate of Technical Substitution change? If I change the ratio how does it change the substitution? **This is going to give us a measure of the curvature of the isoquant**.

The value of sigma is always going to be positive.

The Rate of Technical Substitution and the ratio pair are always going to move in the same direction as long as we assume that the isoquants are concave wrt the origin. If they are not than all bets are off.

Analysis:

$$\sigma = \frac{\%\Delta(k/\mathbf{l})}{\%\Delta RTS} = \frac{d(k/\mathbf{l})}{dRTS} \cdot \frac{RTS}{k/\mathbf{l}} = \frac{\partial \ln(k/\mathbf{l})}{\partial \ln RTS}$$

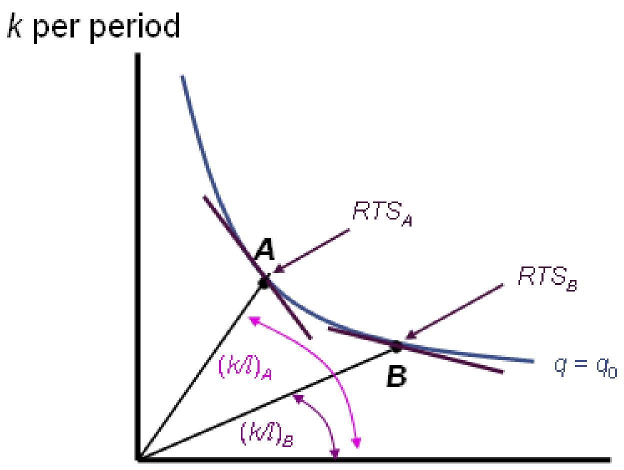
We start off with a ratio of k/l, we decrease it, how does our rate of technical substitution change? This point is greater, the rate of technical substitution is greater, as we decline isoquant smoothes out. That's what elasticity of substitution measures. It measures the curvature of that isoquant line. So is sigma is high RTS is not going to change much relative to a change in the ratio of the inputs. This means the isoquant is relatively flat. We would be in the high levels of labor area.

If your in the high levels of capital then the isoquant is going to be sharply curved and the sigma is going to be lower in those points. We are saying that it is certainly possible for sigma to change on an isoquant, in fact this will almost always be the case.

(basically saying that k/l is a function of RTS, can be solved that way then the derivative taken)

Elasticity of Substitution, cont'd

- Both RTS and k/l will change as we move from point A to point B



σ is the ratio of these proportional changes

σ measures the curvature of the isoquant

Elasticity of Substitution, cont'd

- If σ is high, the RTS will not change much relative to k/l
  - the isoquant will be relatively flat
- If σ is low, the RTS will change by a substantial amount as k/l changes
  - the isoquant will be sharply curved
- It is possible for σ to change along an isoquant or as the scale of production changes

Optimizing Behavior

- Constrained Cost Minimization

- ∗ The economic cost of any input is the payment required to keep that input in its present employment
  - The remuneration the input would receive in its best alternative employment
- ∗ Total costs for the firm are given by

$$\text{total costs} = C = wl + vk$$

Different behaviors the firm will use to optimize, minimize cost for example. Many different ways. This example is a constrained cost minimization.

If I want to produce a certain level of output, say 1000 widgets, what is the optimal mix of capital and labor I should put forth? What is the lowest cost to produce a fixed number of widgets. But this is not the only question we can answer. We can answer the question …

Constrained Cost Minimization

❑ Cost-Minimizing Input Choices

- ∗ To minimize the cost of producing a given level of output, a firm should choose a point on the isoquant at which the **RTS** is equal to the **ratio w/v**
- ∗ It should equate the rate at which k can be traded for l in the productive process to the rate at which they can be traded in the marketplace

Constrained output maximization. Suppose we have a cost budget, say $10,000 to spend any way we like on a mixture of capital and labor. Given that fixed level of cost, the question becomes, **given that fixed budget how do I maximize output?**

We will first look at constrained cost minimization. If I want to produce a certain level of widgets what is the lowest cost required. Then we will look at a maximization problem that says how many widgets do I need to produce and how do I need to manufacture them in order to maximize my profits.

So we see there are 3 or 4 maximization problems we can look at in this area.

Total Cost of a Firm:

$$\textbf{total costs} = \mathbf{C} = \mathbf{wl} + \mathbf{vk}$$

w: labor rate, cost per something such as employee
v: cost of capital
k: capital, cost I need to put into the business.
L: labor
Total cost is the number of units of each of the cost that I employee multiplied by the cost per unit. (w and v are like the cost of l and k).

So to minimize the cost of producing a given level of output how do I get to that optimal point. We will change the mix of labor and capital so that the rate of technical substitution, RTS, is equal to the ratio of the cost of either input. Similar to what we did in last class. We will see that the optimal point is the mix of v and w which is tangential to the isoquant.

Cost-Minimizing Input Choices, cont'd

- Mathematically, we seek to minimize total costs given $q = f(k,l) = q_0$
- Setting up the Lagrangian:

$$L = wl + vk + \lambda[q_0 - f(k,l)]$$

- First order conditions are

$$\partial L/\partial l = w - \lambda(\partial f/\partial l) = 0$$

$$\partial L/\partial k = v - \lambda(\partial f/\partial k) = 0$$

$$\partial L/\partial \lambda = q_0 - f(k,l) = 0$$

We solve this problem using Lagrangian maximization for fixed q (output to manufacture). Cost function, price of labor times labor units plus price of capital times capital units. Budget constraint in this case is the number of units I need to produce minus the production function. Take the partials listed above, set them equal to 0.

Solve each of the above partials for lamda, then set the equations equal (via lamda) and solve for W/v (ratio of the input prices):

$$\text{Notation: } \frac{\delta f}{\delta l} = f_l \qquad \frac{\delta f}{\delta k} = f_k \qquad \text{then ...}$$

$$\lambda = \frac{w}{f_l} \qquad \lambda = \frac{v}{f_k} \qquad \text{RTS} = \frac{w}{v} = \frac{f_l}{f_k}$$

If I can give labor and take on more capital, given the prices I have to pay for both to make that economically sensible decision, then I will carry on doing that until the ratio of prices in the market place are equal to the ratio of the utilization in my production equation.

Cost-Minimizing Input Choices, cont'd

- Dividing the first two conditions we get

$$\frac{w}{v} = \frac{\partial f / \partial \mathbf{l}}{\partial f / \partial k} = RTS \ (\mathbf{l} \text{ for } k)$$

- The cost-minimizing firm should equate the *RTS* for the two inputs to the ratio of their prices

To minimize cost set the RTS equal to the ratio of market prices for the two units (capital and labor). That will optimize capital.

We can also think about it in these terms…

Cost-Minimizing Input Choices, cont'd

- Cross-multiplying, we get

$$\frac{f_k}{v} = \frac{f_{\mathbf{l}}}{w}$$

- For costs to be minimized, the marginal productivity per dollar spent should be the same for all inputs

$\frac{f_k}{v}$ is the rate of change of my production by increasing capital, cost of capital.

$\frac{f_l}{w}$ is the rate of change of my production by increasing labor, cost of labor.

$\frac{f_k}{v} = \frac{f_l}{w}$, marginal productivity per dollar spent is going to be the same. Why? Well if I have a choice between spending a dollar on labor or a dollar on capital, and there is a trade off between the two, I'm going to spend it on the input that gives me the biggest return on dollar spent. I'm going to keep making that exchange until the two ratios descend (?).

This is telling me that the return I'm getting per dollar of capital is the same as the return I'm getting on labor per dollar of labor. If this ration equality is out of balance then I'm going to change the mixture of labor and capital to bring it back into balance.

This is telling me if I increase my spending on capital this is the increase in output I get given a change in capital. This is the increase in productivity per dollar of capital set equal to the increase in productivity per dollar of labor per dollar of labor. Therefore **I'm going to change the mix of capital and labor along an isoquant until these two ratios are set equal**. **If one is higher than the other I'm going to spend more on the higher one and less on the lower productivity one**.

I'm going to change my position on the isoquant until these two ratios are set equal.

Cost-Minimizing Input Choices, cont'd

- Note that this equation's inverse is also of interest

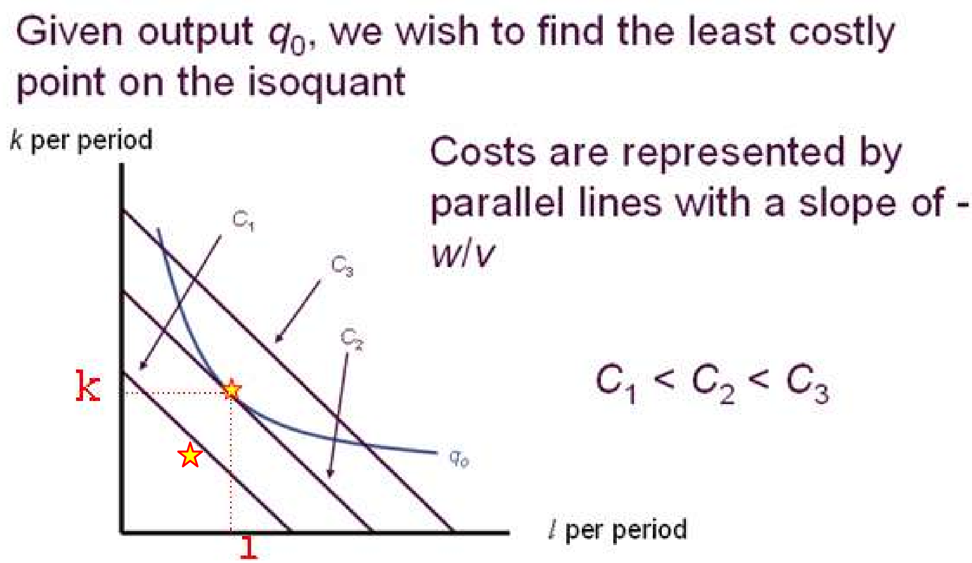
$$\frac{w}{f_l} = \frac{v}{f_k} = \lambda$$

- The Lagrangian multiplier shows how much in extra costs would be incurred by increasing the output constraint slightly.

In this case the Lagrangian Multiplier is telling us how much extra cost I will incur by increasing the number of units. My constraint in this problem is the number of units I have to produce are fixed. In the maximization of utility my budget is fixed. So what this is telling me is **if I can increase the number of units produced by a fixed amount (say one more unit) the shadow price tells us how much the costs need to increase to produce that one extra unit**. Very useful, it avoids our having to solve two maximization problems.

This will give us an approximate idea of how much additional costs I need to produce one additional unit (it's only APPROXIMATE).

Cost-Minimizing Input Choices , cont'd

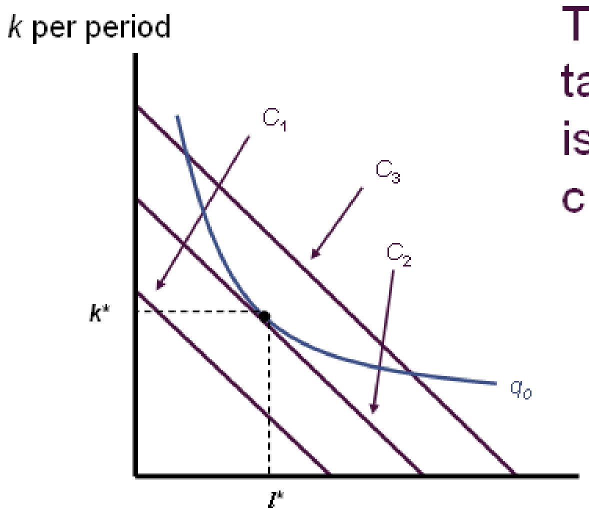


We want to produce a certain number of widgets, q_0. The most efficient way of my producing the certain number of widgets is given by the level of labor and capital where the rate of technical substitution, RTS, is equal to the ratio of prices of labor and capital. In this case the tangent of q_0 at the intersection of isoquant q_0 and c_2 (at the ☆).

So the minimum way of producing q_0 is to manufacture at the c_2 at the mix of l and k indicated by the star, this will give optimal production, minimum cost. (see next page)

Cost-Minimizing Input Choices , cont'd

The minimum cost of producing q_0 is C_2



This occurs at the tangency between the isoquant and the total cost curve

The optimal choice is l^*, k^*

Input Demands

- Input Demand Functions
  - Cost minimization leads to a demand for capital and labor that is contingent on the level of output being produced
  - The demand for an input is a derived demand
    - It is based on the level of the firm's output

In the same way that we can construct demand functions in a utility sense, we can construct input demand functions. Given some level of output, I can find the minimized cost by mixing the appropriate units of capital and labor. If I want to produce so many widgets I can do so at the lowest cost by finding the optimal level of capital and labor to use, where the rate of technical substitution, RTS, is equal to the market ratio of prices.

The Firm's Expansion Path

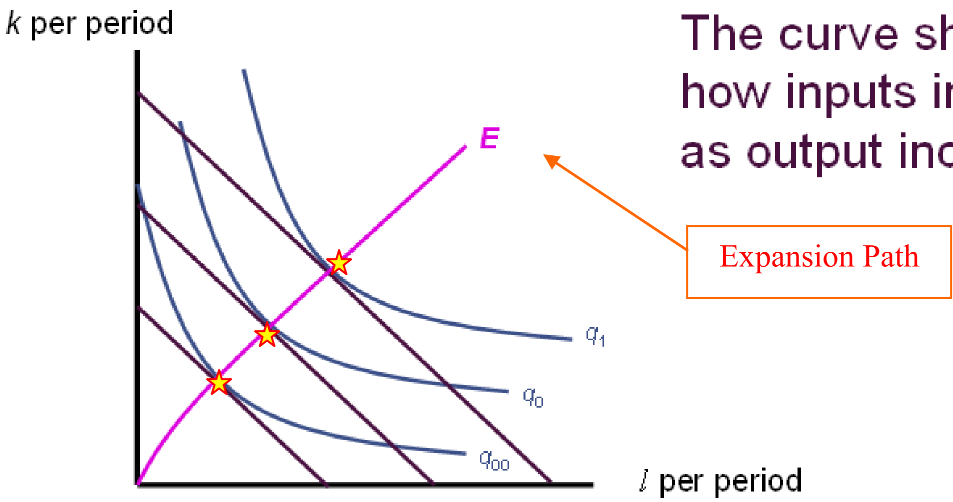
- The firm can determine the cost-minimizing combinations of k and l for every level of output
- If input costs remain constant for all amounts of k and l the firm may demand, we can trace the locus of cost-minimizing choices
  - Called the firm's expansion path

If we do this, given a different level of output I can solve the problem over and over. So I can actually construct, for a given level of output, how much labor or capital I'll use. I can construct, for the firm, input demand functions. Given a level of output I can tell you the optimal level of capital needed to generate that output at the lowest cost to me and the firm.

For a given level of output we are tracing the point which minimizes the cost. So given a level of output, such as q_3, we can tell you the level of labor and capital needed to solve the minimization problem. (see graph below)

The Firm's Expansion Path

The expansion path is the locus of cost-minimizing tangencies



The curve shows how inputs increase as output increases

- ❑ The expansion path does not have to be a straight line
  - ∗ The use of some inputs may increase faster than others as output expands
    - • Depends on the shape of the isoquants
- ❑ The expansion path does not have to be upward sloping
  - ∗ If the use of an input falls as output expands, that input is an inferior input

Two Step Process:
1) Given output level
2) Find minimum cost by finding the ratio of capital and labor I need to minimize my cost given the level of output.

The line which shows me the optimal mix of capital and labor in the above case, we call that he firms **Expansion Path**. As the firm ramps up production, the expansion path tell me the mix of capital and labor needed at any level of production. It's the optimal mix of capital and labor.

How do we find it? We trace out the cost minimization choice of capital and labor. Given a level of production I'll find a minimized cost which will tell me the mixture of capital and labor, that point of capital and labor is going to be the expansion path point for that isoquant / capital/labor pair. Connecting each expansion path point gives the expansion path. The points progressing outward represent the progressively larger isoquant / capital-labor pairs, the northeast direction shows the firm is expanding along these points, increasing it's output. And this doesn't necessarily have to be a straight line and it almost certainly won't be a straight line. As I get more and more production, I may need lessor quantities of capital relative to labor. I may need to increase the labor wrt the point (?). So there is no reason way the path of expansion has to be a straight line. The relative quantities of the two inputs may change. What will determine that is the shape of the isoquants. So these will have to be very specialized curves in order for the expansion path to be a straight line. It depends on the production and technology of the firm.

Total Cost Function

- The total cost function shows that for any set of input costs and for any output level, the minimum cost incurred by the firm is
$$C = C(v,w,q)$$
- As output (q) increases, total costs increase

We can also talk about the total cost function for a particular company. What will it look like? If I have an output level, I can solve for the best mix of labor and capital I need to minimize my cost. Therefore I can construct cost function that tells me what my cost is going to be to produce any level of output. This will be a function of my production function (the shape of my isoquants) and also a function of the relative prices of the input functions.

So it's going to be dependent on the shape of the isoquants, and also dependent on the slope of the costs lines (k/l). So the prices of the factors, the cost of labor and the cost of capital) are going to be important. If one factor increases it's cost, then I am going to have to rebalance my inputs in order to equate the RTS to the new ratio of prices. I'm going to have to resolve this problem. If this line suddenly shifts because one of the input cost goes up, then I'm going to have to resolve the minimization of those costs, I'm going to have to reset the tangent of that line equal to the new ratio of prices. It is a dynamic process.

As output increases total cost is certainly going to increase, assuming we don't have any negative return to any of our factors. As we focus on the cost function there are a couple of ratios we are going to be interested in. The first one is the Average Cost Function…

Average Cost Function

- The average cost function (AC) is found by computing total costs per unit of output

$$\text{average cost} = AC(v, w, q) = \frac{C(v, w, q)}{q}$$

We've moved on now from looking at labor and capital in terms of units of input. We are now looking at cost in total dollar terms. **Given the output that we need we can find the optimal mix of labor and capital**. If I know the prices I can tell you what the total cost is going to be to manufacture that level output. Now when we are looking at the cost function we are truly looking at dollars worth of cost.

Average cost: if I look at the total costs of the optimal production level, of the optimal mix of capital and labor, then I can compare total cost to produce so many units to how many units are being produced. Can calculate the average cost in dollar terms per unit widgets produced.

Average cost = total costs / number of units

Marginal Cost Function

- The marginal cost function (MC) is found by computing the change in total costs for a change in output produced

$$\text{marginal cost} = MC(v, w, q) = \frac{\partial C(v, w, q)}{\partial q}$$

Cost is a function of input prices and units of production. Therefore to find the rate of change of cost wrt units of output it is simply the partial derivative of cost wrt units produced. Taking the derivative of cost with respect to quantity holding all other variables constant and assuming the price is going to change. Consider what this will look like graphically…

Graphical Analysis of Total Costs

- Suppose that k_1 units of capital and l_1 units of labor input are required to produce one unit of output

$$C(q = 1) = vk_1 + wl_1$$

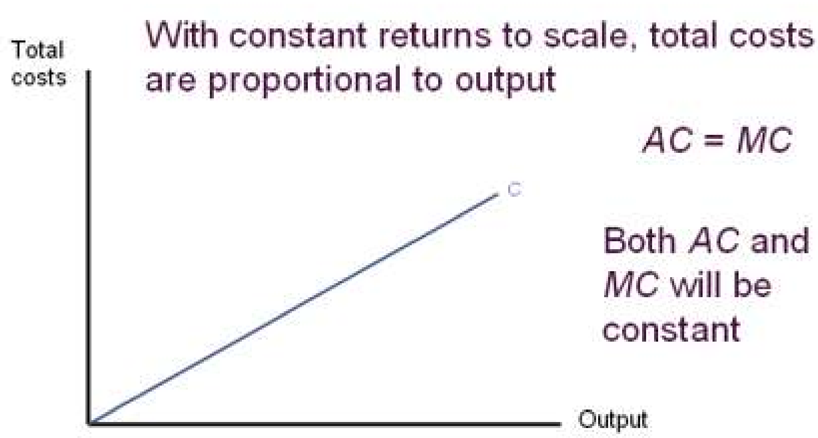
- To produce m units of output (assuming constant returns to scale)

$$C(q = m) = vmk_1 + wml_1 = m(vk_1 + wl_1)$$
$$C(q = m) = m \cdot C(q = 1)$$

Suppose I am going to produce 1 unit of output. To do this I need k_1 units of capital and l_1 units of labor, those are the optimal mix that I need to produce 1 widget. Therefore the cost of producing 1 widget is going to be given to me by cost of capital times the number of units of capital plus the cost of labor times the number of units of labor. Be clear, **this <u>level of capital</u> and this <u>level of labor</u> comes from the maximization solving problem** (a few pages back). I could use different ratios of k and l to produce the same number of outputs, but I'm interested here in producing to the optimal ratio.

Now say I need to produce m units. Assuming constant returns to scale, in order to produce m units I'm going to use m units of labor and m units of capital, **<u>if I have constant returns to scale</u>**. If I do have constant returns to scale my total costs is a straight line.

Graphical Analysis of Total Costs

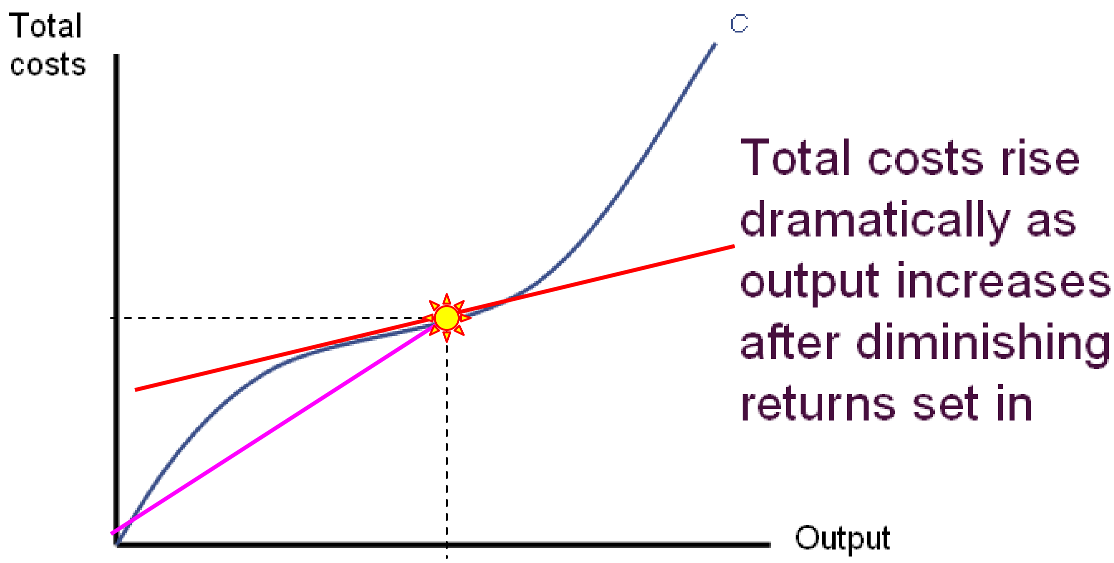


If I do have constant returns to scale my total costs is a straight line. As output goes up cost goes up, there is a linear relationship between the two. What's more, **<u>my average cost is equal to my marginal cost</u>**. Why? Well average cost is simply the slope of this line (since it's straight) and marginal cost is also the slope of this line because it's the rate of change. In this case both AC and MC will be constant.

IF I HAVE CONSTANT RETURN TO SCALE <u>AVERAGE COST</u> IS EQUAL TO <u>MARGINAL COST</u>

Graphical Analysis of Total Costs, cont'd

- Suppose instead that total costs start out as concave and then becomes convex as output increases
  - **One possible explanation for this is that there is a third factor of production that is fixed as capital and labor usage expands**
  - Total costs begin rising rapidly after diminishing returns set in



Initially it takes a lot of cost to produce a small number of units. There are a lot of fixed costs say. Costs jumps at the one or two units level (small q in general). At this stage there is not much efficiency. After that I hit a sweet spot where I add additional cost and produce additional units. After that inefficiencies start to creep in, just become too big. Think of it in terms of a factory, if I start off producing widgets I need a certain level of additional factory floor, weather I'm producing one or 10 widgets.

Eventually that same labor force is able to produce 100 or even 1000 units. But if I want to produce 10,000 units I have to put an extra shift on, I start to see inefficiencies. So initially it is very costly per unit, then sweet spot, then after that very costly again because inefficiencies start to creep into the system. So our curve has kind of an S shape.

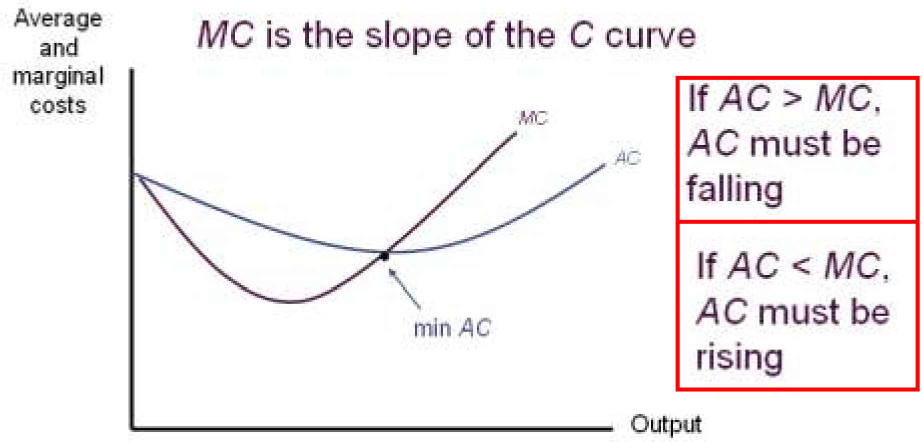
Another component of this behavior may be a third factor which we are ignoring or pushing into the background but which may be producing different effects on the cost structure. This is plausible.

Now what's this going to look like? Pick a point (above). **Average Cost**, it takes me so many dollars to produce that many units, the average cost is the slope of the line joining the origin and the production point. **Marginal Costs** is the tangential point to the slope of the curve at the point of production.

Consider the slope, starts steep, flattens out, becomes steep again. Initially the average cost is going to be lower than the marginal costs, the line starts off steep. But eventually the marginal costs is less than the average cost. It reverses. The marginal costs is going to be greater than the average costs.

WHAT WILL THIS LOOK LIKE IF I MAP A GRAPH OF AVERAGE COSTS AND MARGINAL COSTS?

Graphical Analysis of Total Costs, cont'd



Marginal costs starts high, diminishes, and then increases. Average costs, starts off at a certain level, diminishes, eventually the inefficiencies give a knee up to average cost (?) and we start to see average cost increasing. Notice if average costs are greater than marginal costs then average costs must be falling. Visa-versa, see graph above. In fact, if you look at the point of intersection of these two lines, that's where marginal costs overtake average costs.

When we talk about monopolies this point of intersection will have a huge impact on monopolies behavior. As long as markets are not completely competitive there is going to be an impact of marginal price versus marginal cost. That will give us the answer.

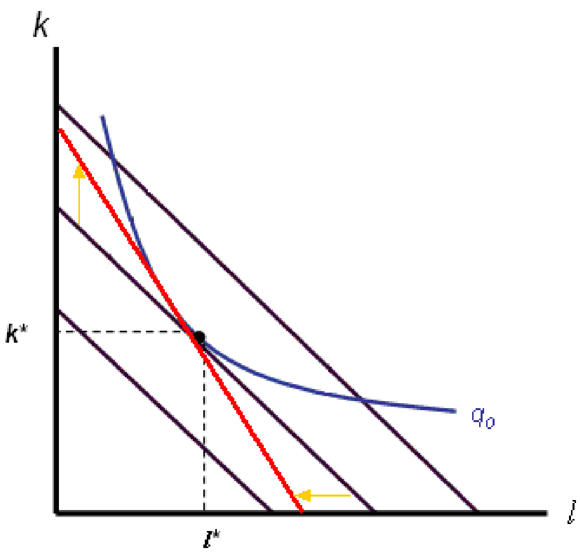
Input Substitution

- A change in the price of an input will cause the firm to alter its input mix
- We wish to see how **k/l** changes in response to a change in w/v, while **holding q constant**

$$\frac{\partial\left(\frac{k}{\mathbf{l}}\right)}{\partial\left(\frac{w}{v}\right)}$$

Here we are talking about prices of the inputs, the prices w and v of capital and labor. Suppose the ratio of the prices (w/v) change. On my maximization line the slope of the straight line, the trade off between the cost of capital and labor, changes because the price relative to capital and labor changes.

What we are interested in is seeing how our optimal mix of capital and labor changes because the relative price of those two goods changes. We can see in the graph if the ratio of the prices in capital and labor shift then it's going to lead us to a different mix of capital and labor. If we are trying to minimize cost for a certain level of output the way we do this is set the ratio of technical substitution (RTS) equal to the ratio of prices. If the ratio of prices changes then it's going to lead to a different mix of capital and labor. Will lead to a different outcome when we do the maximization problem.



So what we are interested in is seeing how much our mix changes given a change in prices. So a change in the ratio of prices, how that impacts the change in the relative portions of capital and labor. This is what is meant by **RATIO OF INPUT SUBSTITUTION**.

Input Substitution

- Putting this in proportional terms as

$$s = \frac{\partial(k/\mathbf{l})}{\partial(w/v)} \cdot \frac{w/v}{k/\mathbf{l}} = \frac{\partial \ln(k/\mathbf{l})}{\partial \ln(w/v)}$$

gives an alternative definition of the elasticity of substitution

- **In the two-input case, *s* <u>must</u> be nonnegative** (due to idea that ratio of one thing goes up you will start using the other input more).
- Large values of s indicate that **firms change their input mix significantly** if input prices change (a change in prices is going to lead to a dramatic change in the level of inputs)

$$\frac{w}{v} \rightarrow \frac{w}{v} + \Delta(\frac{w}{v})$$

$$\frac{k}{l} \rightarrow \frac{k}{l} + \Delta(\frac{k}{l})$$

The two ratios undergo a small percentage change. The ratio of our input capital to labor given our initial capital to labor. This ratio is the percentage change. So we are adding the percentage change of each respective ratio to the initial proportion.

What we are interested in is this, if the ratio of labor cost to capital cost increases by a small positive amount, that means the labor costs go up relative to capital costs, by some small amount. That ratio changes by a very small amount. Now what is going to happen to the mix of capital and labor? Well this will lead to a change in capital and labor. Notice how these ratios work, if the cost of labor goes up relative to the coast of capital then we will have a small positive increase in this ratio. **This is the amount of capital compared to the amount of labor. So if this increases by a small amount you would expect capital to be substituted for labor. <u>Labor is becoming more expensive relative to capital</u>. Therefore you would think that capital should increase relative to labor (to offset the need for more expensive labor).** A small increase in this ratio should lead to a small increase in this ratio.

So we are interested in the percentage change, so the small change in capital over labor divided by the amount of capital over labor (the percentage change in that ratio). Divided by the percentage change in the relative prices. If that ratio changes by 1%, what percentage increase do we see in the mix of capital and labor that we actually use? In the extreme this is what we get when we take it to the limit.

Partial Elasticity of Substitution

- The partial elasticity of substitution between two inputs (x_i and x_j) with prices w_i and w_j is given by

$$s_{ij} = \frac{\partial(x_i / x_j)}{\partial(w_j / w_i)} \cdot \frac{w_j / w_i}{x_i / x_j} = \frac{\partial \ln(x_i / x_j)}{\partial \ln(w_j / w_i)}$$

- S_{ij} is a more flexible concept than σ because it allows the firm to alter the usage of inputs other than x_i and x_j when input prices change

Here we are extending this model to look at more general setups where we have more than two inputs. **In this case the s does not necessarily have to be positive.**

Short-Run, Long-Run Distinction

- In the short run, economic actors have only limited flexibility in their actions
- Assume that the capital input is held constant at k_1 and the firm is free to vary only its labor input
- The production function becomes

$$q = f(k_1, l)$$

We've been talking about how businesses can change the mix of inputs given a certain level of output that they need. But we have to think about the distinction between short run and long run. What we mean is If we tell a company to produce a certain level of units, if I suddenly change their target number of units, they may not be able to increase their factor inputs in an optimal way. Some of their factor inputs may be relatively fixed over the short term.

Example, is a factory is manufacturing widgets and they have a budget to manufacture 100,000 of these widgets, if I suddenly ask them to manufacture 120,000 widgets it may take them a long time to increase plant equipment and capital, they may not be able to do so immediately. They may solve the problem by using more labor. They may do that even though that new mix of labor and capital is not efficient. The reason they do this is because they have a certain target to hit and because they cannot vary one of their inputs they do a "second best." They move away from the isoquant that they would like to be on and over the short term they are willing not to produce at the optimal level.

Over the long run we can change our levels of capital and labor to the point where we need them. So notice the distinction between short run and long run.

Short run, economic actors, firms have only limited flexibility in their actions. They may find it easier to change one input where the other may be less flexible. It may be easier to add one shift of workers than it is to add new capital to the business (such as increase the shop floor).

So ought to assume in the short run that one of the variables is held constant. We usually assume capital is the one held constant in short run cases and the firm is free to vary it's labor only. It can hire new employees and pay them overtime. In this case the production function becomes simply a function of labor, capital remains fixed. **It becomes a single variable function**. Does this change what the firm does? YES. Because short run total costs to the firm are going to be equal to a fixed part and a variable part. Over the short run the cost of my capital times the units of capital are relatively fixed. Therefore my cost function becomes a fixed variable of labor and a fixed cost of capital.

Short-Run Total Costs

- Short-run total cost for the firm is

$$SC = vk_1 + wl$$

- There are two types of short-run costs:
  - short-run fixed costs are costs associated with fixed inputs (vk_1)
  - short-run variable costs are costs associated with variable inputs (wl)

In the short run we usually have fixed costs and variable costs. For example, in accounting we think of costs being fixed and variable. Most costs, over the long run, are variable. You can always build that extra factory. Whereas over the short run, some of the costs are constrained, they cannot be increased very easily. And other costs are relatively variable.

Short-Run Total Costs

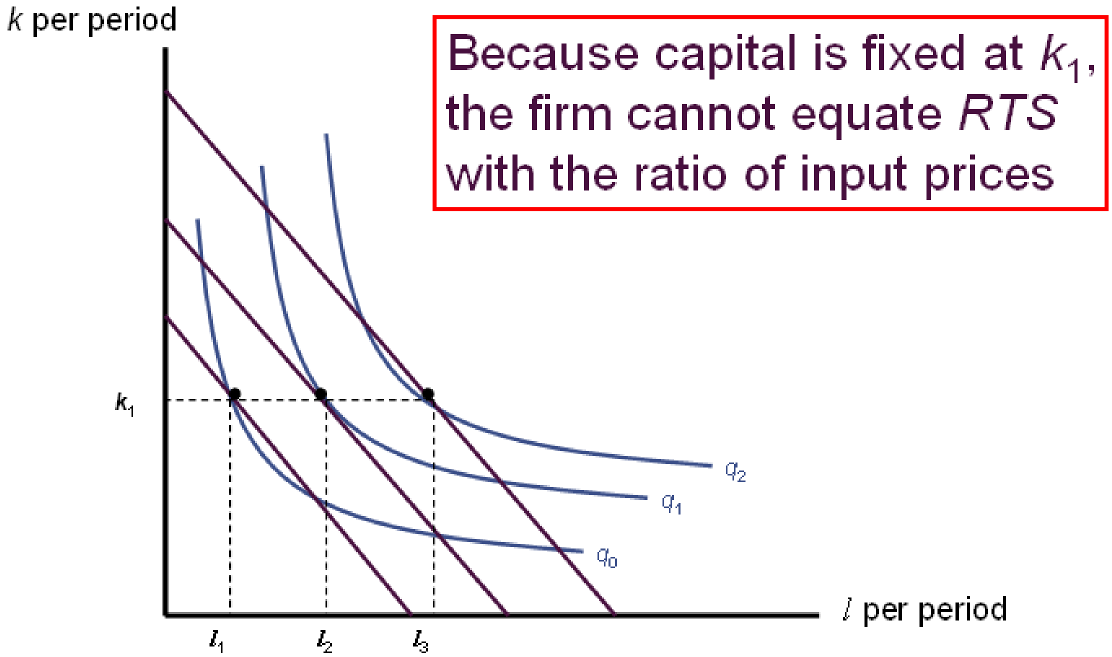
- Short-run costs are not minimal costs that we need for producing the various output levels
  - The firm does not have the flexibility of input choice
  - To vary its output in the short run, the firm must use non-optimal input combinations
  - The RTS will not be equal to the ratio of input prices

If we are constrained on one of our costs we are going to make a second best attempt to produce the number of units that we need. Ideally we would vary both inputs. If one input is fixed we would have to be very lucky to end up at the optimal level of output.

It is more than likely that we are going to be spending more costs than we need we would need to if we were given free long run choices to make. So we will use non-optimal input combinations. When we do so our RTS is very unlikely to be equal to the ratio of input prices.

Physically what does this mean? Lets suppose our capital is fixed. We can't change the level of capital. Given a different level of output and the fixed level of capital, there is only one cost line which will be tangential. And that will be at that particular point. If by some fluke we are asked to output at this level, then this is optimal (but this is unlikely). Otherwise we are forced to use too much capital or too little capital. Flexibility comes into the system because we can change the amount of labor that we need.

Short-Run Total Costs



Because capital is fixed at k_1 we are not able to get to the optimal point on the isoquant. We would like to be at the optimal point but if we are forced to change our production but we cannot change our capital, then we have to make a second best choice. And we can see that in either case we are not at the best production level. The optimal points would have lower costs given the same level of production.

In the above graph l_1 uses too much capital (k) and l_3 uses too little capital. l_2 is about right but that is the point we are being driven off of due to the changed production demand.

[End of Lecture] He will post a practice exam. Monday he will answer any practice exam questions. Everything after midterm. Little on rational/efficient markets, and the last three classes. The homework is a little more involved than the exam will be, less math more interpretation.

Optimizing Behavior

- Profit Maximization
- A profit-maximizing firm chooses both its inputs and its outputs with the sole goal of achieving maximum economic profits
  - Seeks to maximize the difference between total revenue and total economic costs

Profit Maximization, cont'd

- If firms are strictly profit maximizers, they will make decisions in a "marginal" way
  - Examine the marginal profit obtainable from producing one more unit of hiring one additional laborer

Output Choice

- Total revenue for a firm is given by

$$R(q) = p(q) \cdot q$$

- In the production of q, certain economic costs are incurred [C(q)]
- Economic profits (π) are the difference between total revenue and total costs

$$\pi(q) = R(q) - C(q) = p(q) \cdot q - C(q)$$

Output Choice

- The necessary condition for choosing the level of q that maximizes profits can be found by setting the derivative of the π function with respect to q equal to zero

$$\frac{d\pi}{dq} = \pi'(q) = \frac{dR}{dq} - \frac{dC}{dq} = 0 \qquad \frac{dR}{dq} = \frac{dC}{dq}$$

Output Choice

- To maximize economic profits, the firm should choose the output for which marginal revenue is equal to marginal cost

$$MR = \frac{dR}{dq} = \frac{dC}{dq} = MC$$

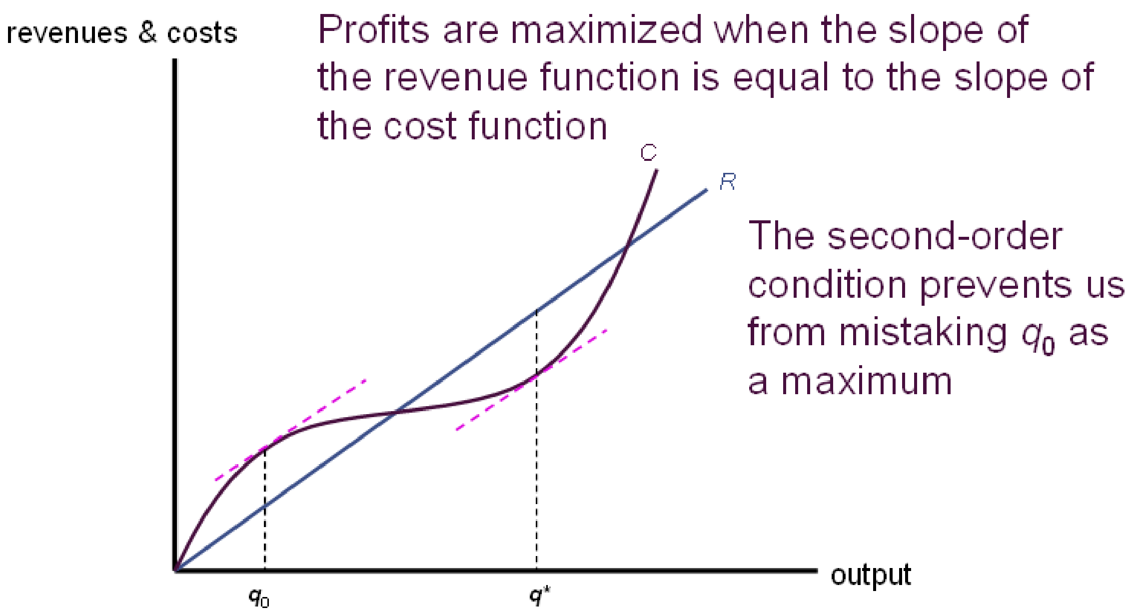
Second-Order Conditions

- MR = MC is only a necessary condition for profit maximization
- For sufficiency, it is also required that

$$\left.\frac{d^2\pi}{dq^2}\right|_{q=q^*} = \left.\frac{d\pi'(q)}{dq}\right|_{q=q^*} < 0$$

∗ “marginal” profit must be decreasing at the optimal level of q

Profit Maximization



Marginal Revenue

- If a firm can sell all it wishes without having any effect on market price, marginal revenue will be equal to price
- If a firm faces a downward-sloping demand curve, more output can only be sold if the firm reduces the good’s price

$$\text{marginal revenue} = MR(q) = \frac{dR}{dq} = \frac{d[p(q)\cdot q]}{dq} = p + q\cdot\frac{dp}{dq}$$

Marginal Revenue

- If a firm faces a downward-sloping demand curve, marginal revenue will be a function of output
- If price falls as a firm increases output, marginal revenue will be less than price

Marginal Revenue

- Suppose that the demand curve for a sub sandwich is

$$q = 100 - 10p$$

- Solving for price, we get

$$p = -q/10 + 10$$

- This means that total revenue is

$$R = pq = -q^2/10 + 10q$$

- Marginal revenue will be given by

$$MR = dR/dq = -q/5 + 10$$

Profit Maximization

- To determine the profit-maximizing output, we must know the firm's costs
- If subs can be produced at a constant average and marginal cost of $4, then

$$\text{MR} = \text{MC}$$
$$-q/5 + 10 = 4$$
$$q = 30$$

Profit Maximization and Input Demand

- A firm's output is determined by the amount of inputs it chooses to employ
  - the relationship between inputs and outputs is summarized by the production function
- A firm's economic profit can also be expressed as a function of inputs

$$\pi(k,l) = pq - C(q) = pf(k,l) - (vk + wl)$$

Profit Maximization and Input Demand

- The first-order conditions for a maximum are

$$\partial\pi/\partial k = p[\partial f/\partial k] - v = 0$$
$$\partial\pi/\partial l = p[\partial f/\partial l] - w = 0$$

- A profit-maximizing firm should hire any input up to the point at which its marginal contribution to revenues is equal to the marginal cost of hiring the input

Profit Maximization and Input Demand

- These first-order conditions for profit maximization also imply cost minimization
  - they imply that RTS = w/v

Profit Maximization and Input Demand

- To ensure a true maximum, second-order conditions require that

$$\pi_{kk} = f_{kk} < 0$$
$$\pi_{ll} = f_{ll} < 0$$
$$\pi_{kk}\pi_{ll} - \pi_{kl}^2 = f_{kk}f_{ll} - f_{kl}^2 > 0$$

  - capital and labor must exhibit sufficiently diminishing marginal productivities so that marginal costs rise as output expands